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FLUCTUATIONS OF A RENEWAL-REWARD PROCESS

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ABSTRACT

Fluctuation theory is concerned with the study of extreme values of sums of independent, arbitrary-valued random variables. Simple but powerful combinatorial methods due chiefly to E. S. Andersen, F. Spitzer, and W. Feller have recently provided an easy method of attack on these problems. However, operations research models are concerned with fluctuations of various economic returns which are earned at random points in time, and whose increments are correlated with the interval since the last payoff. Our generalization considers the fluctuations of a cumulative reward process, defined on an underlying renewal process. Most of the classical results carry through, including Weiner-Hopf type factorization, an Andersen-Pollaczek-Spitzer type identity, and certain Waldian-Pollaczek results. As applications, we find the distribution of the maximum return over a mixed index-epoch horizon, and show how certain general results for the GI/G/1 queue follow directly from the various three-dimensional ladder distributions.

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1. INTRODUCTION

Fluctuation theory is concerned with the extreme values of the sums

$$\rho_0 = 0 \; ; \; \rho_n = \sum_{i=1}^{n} \xi_i \qquad (n = 1,2, \ldots) \tag{1}$$

where the ξ_i are independent random variables with common distribution Q. In the case where the ξ_i are nonnegative, we have a simple problem in *renewal theory*. However, when arbitrary values of the ξ_i are allowed, the theory becomes more difficult; it is only recently that simple, but powerful, combinatorial methods, developed by E. S. Andersen [1,2,3], and extended and simplified by F. Spitzer [17,18,19,20], and W. Feller [9], have been discovered for the general case. Readable summaries of this modern approach and more detailed references may be found in Chapters 12 and 18 of [8], [14], and Chapter 6 of [16].

An interesting generalization results if one thinks of the ρ_n in (1) as a *cumulative*, or *reward* process, "earned" at *epochs* in discrete time, $n = 1,2, \ldots$; we then allow these reward epochs to take on a random nature by associating them with *the epochs of a renewal process*. Thus, our generalization is the study of the *fluctuations of a reward process imbedded in a renewal process*.

Let the underlying renewal process consist of nonnegative intervals $\{\tau_i\}$, $(i = 1,2, \ldots)$, with common distribution A; then, the location of the n^{th} epoch is at

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(2) $$\mathfrak{s}_0 = 0 \ ; \ \mathfrak{s}_n = \sum_{i=1}^{n} \tau_i \qquad (n = 1,2, \ldots)$$

and the total reward accumulated to that point is defined as:

(3) $$\rho_0 = 0 \qquad \rho_n = \sum_{i=1}^{n} \xi_i(\tau_i)$$

At times between these epochs, the total reward remains constant at its last previous value, i.e., we define:

(4) $$\rho(t) = \rho_{\eta(t)} = \begin{cases} 0 & \eta(t) = 0 \\ \sum_{i=1}^{\eta(t)} \xi_i(\tau_i) & \eta(t) > 0 \end{cases} \quad , \quad (t \geq 0)$$

with

(5) $$\eta(t) = \sup\ (k \mid \mathfrak{s}_k \leq t)$$

as the number of epochs in $[0,t]$. A typical realization of $\rho(t)$ is shown in Figure 1.

Intuitively we may think of $\xi_i(\tau_i)$ as an interval reward, possibly dependent on τ_i , which is "earned" at the end of interval i ; $\rho(t)$ is then the total reward earned in $[0,t]$. For this generalization, we must specify the joint distribution,

(6) $$Q(t;y) = P\{\tau_i \leq t \ ; \ \xi_i \leq y\} \qquad \begin{pmatrix} i = 1,2, \ldots \\ t \geq 0 \\ -\infty < y < -\infty \end{pmatrix}$$

whose marginal distributions are $A(t)$ and $Q(y)$. (We assume $A(0) < 1$, $P\{\xi_i = 0\} < 1$, and $Q(t;y)$ is an honest distribution.)

Since the *imbedded reward process* $\{\rho_n = \rho(\mathfrak{s}_n)\}$ is the same as the fluctuation process of (1), i.e., interval rewards are independent from epoch to epoch, much of

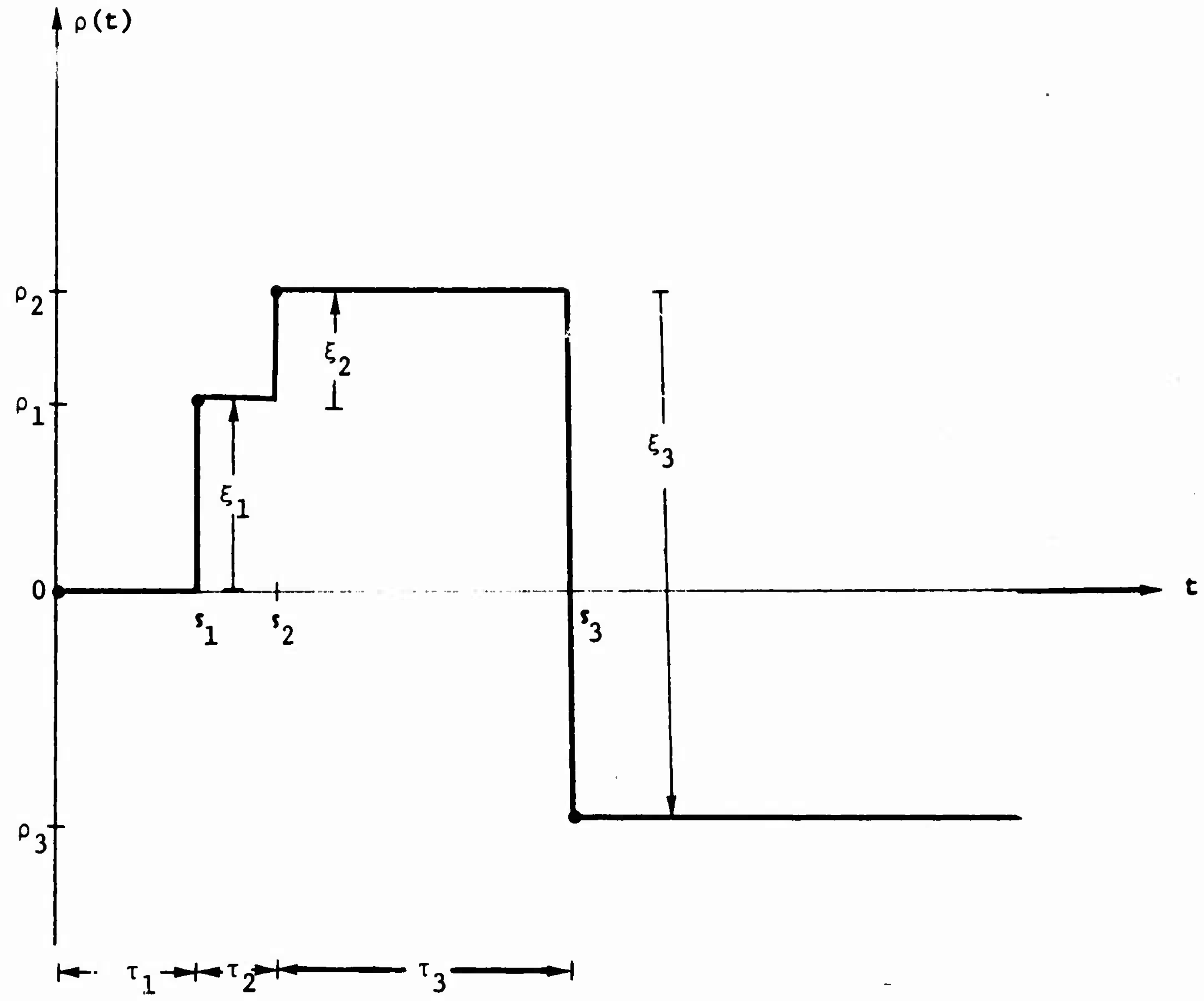

FIGURE 1. TYPICAL REALIZATION OF A (CUMULATIVE) REWARD PROCESS, DEFINED OVER AN UNDERLYING RENEWAL PROCESS.

the simplicity of the original model is retained. However, the addition of the renewal process $\{\mathfrak{s}_n\}$ as a "random clock" leads naturally, as we shall see, to more complex fluctuation models, particularly in queueing theory.

The process $\rho(t)$ could also be thought of as a two-dimensional random walk $(\rho_n, \mathfrak{s}_n)$ in which the second component can only increase. However, this generality will not be needed in what follows.

2. THE IMBEDDED RENEWAL PROCESS: LADDER POINTS

The key to analysis of fluctuation processes lies in a concept first used by D. Blackwell [7]: *the ladder point*. A (strict ascending) ladder point is said to occur at index n and epoch $\mathfrak{s}_n$ iff

$$\rho(\mathfrak{s}_n) > \rho(t) \qquad (0 \le t < \mathfrak{s}_n) \quad ; \tag{7}$$

the *ladder process* obtained by connecting the ladder points in the obvious way is then an *upper envelope*, or *maximum*, to the process $\rho(t)$, (see Figure 2). The origin is sometimes considered as a zero$^{\text{th}}$ ladder point.

We distinguish *three* ladder variables in Figure 2: the *index* at which the record value took place; the *epoch* associated with that index; and the *height* of that record value. For the first (strict ascending) ladder point, we have:

(8) (i) an index $\nu_1 = \min\{n \mid \rho(\mathfrak{s}_n) > 0\} > 0$,

(ii) an epoch $\phi_1 = \mathfrak{s}_{\nu_1} \ge 0$,

and (iii) a height $\lambda_1 = \rho\left(\phi_1^+\right) = \rho_{\nu_1} > 0$,

with joint distribution

$$F(n,t;y) = P\{\nu_1 = n \,,\, \phi_1 \le t \,;\, \lambda_1 \le y\} \quad . \quad \begin{pmatrix} n = 1,2, \ldots \\ 0 \le t < \infty \\ 0 < y < \infty \end{pmatrix} \tag{9}$$

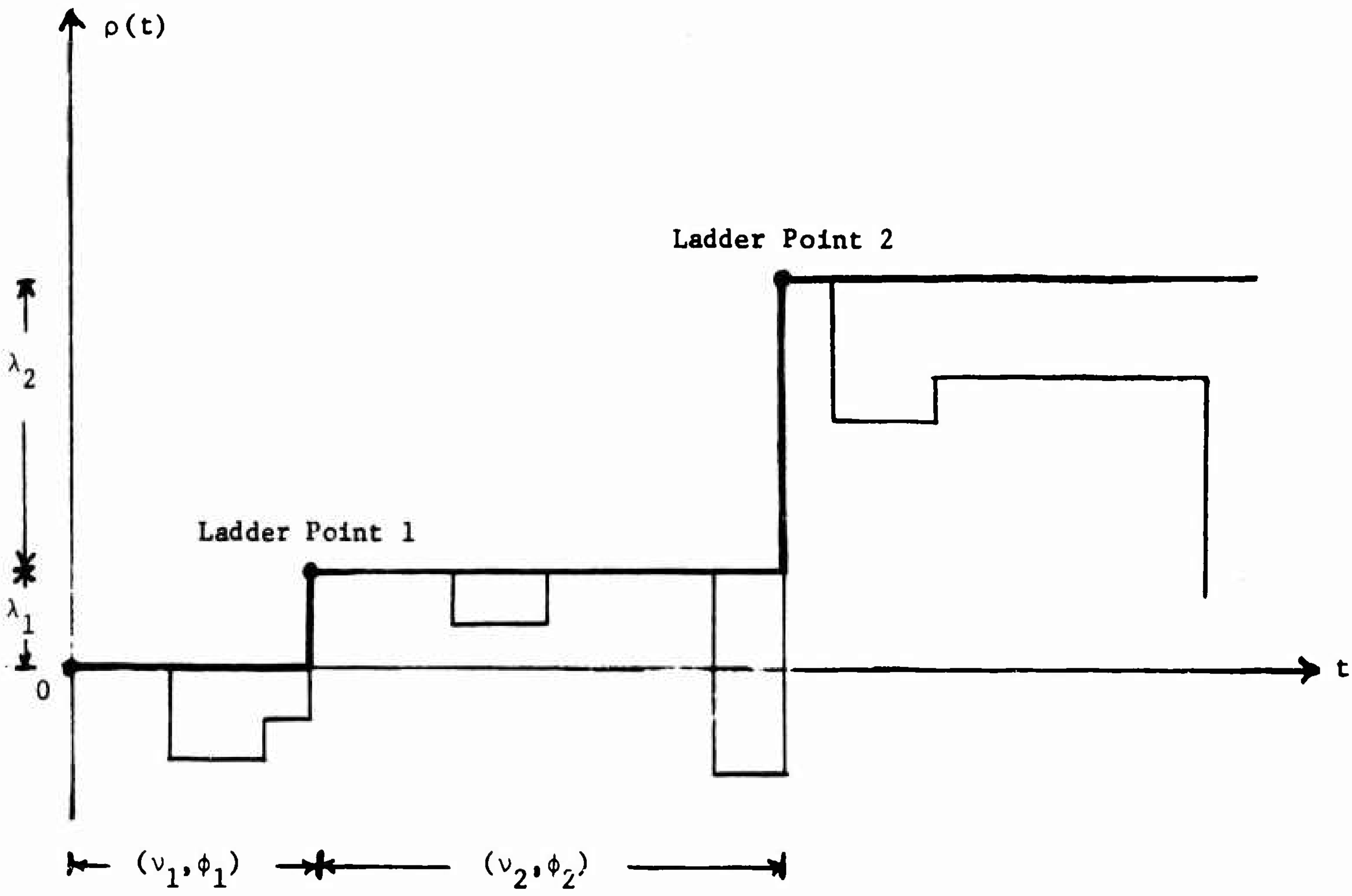

FIGURE 2. REWARD PROCESS SHOWING (STRICT ASCENDING) LADDER POINTS AND LADDER PROCESS, (STRICT ASCENDING) LADDER INDICES, EPOCHS, AND HEIGHTS.

It follows from the definitions and Figure 2 that the distributions of *all* the ladder variables could be calculated from (9), since the ladder points constitute *an imbedded (three-dimensional) renewal process*, in which each new record value occurs "from scratch," starting at the previous value; i.e., the triplet $(\nu_i, \phi_i; \lambda_i)$ is independent and identically distributed *for all* $i = 1,2, \ldots$ An obvious, but important, identity relates the transform of the distribution of the r^{th} ladder point to that of the first ladder point:

$$(10)\qquad E\left\{z^{(\nu_1+\nu_2+\ldots+\nu_r)} \cdot e^{-s(\phi_1+\phi_2+\ldots+\phi_r)} \cdot e^{-p(\lambda_1+\lambda_2+\ldots+\lambda_r)}\right\} = \left[E\left\{z^{\nu_1} e^{-s\phi_1} e^{-p\lambda_1}\right\}\right]^r . \qquad (r = 1,2, \ldots)$$

We shall henceforth drop the index when discussing an arbitrary triplet $(\nu,\phi;\lambda)$.

In the most interesting cases, the ladder process is *defective*, i.e., at some index, further record values cease to occur; the probability of this occuring is the *defect*, $1 - \sum_{n=1}^{\infty} F(n,\infty;\infty)$.

By weakening, reversing, or weakening and reversing the inequality in (7), four different ladder processes can be obtained. To distinguish between them, we shall use four different superscripts on the variables and on the corresponding distribution F :

$$(11)\qquad \begin{array}{lllll} \text{(i)} & \text{strict ascending} & (>) , & F^{+}, & (\lambda^{+} > 0) \\ \text{(ii)} & \text{weak ascending} & (\geq) , & F^{\oplus}, & (\lambda^{\oplus} \geq 0) \\ \text{(iii)} & \text{strict descending} & (<) , & F^{-}, & (\lambda^{-} < 0) \\ \text{(iv)} & \text{weak descending} & (\leq) , & F^{\ominus}, & (\lambda^{\ominus} \leq 0) \end{array}$$

Note the different regions of definition for the ladder height.

We shall see that the processes (i) and (iv), or (ii) and (iii) are, in a certain sense, *duals* of each other; of course the distinction between (i) and (ii),

or (iii) and (iv) vanishes when Q is continuous. A defective weak process implies a defective strong one, and vice versa; however, not both the ascending and the descending processes can be defective.

3. DUALITY: THE REFLECTION PRINCIPLE

Determination of explicit forms for the ladder distributions depends on two fundamental principles. The first of these, whose proof is trivial, is:

The Reflection Principle

(12) *Consider any realization* $(\tau_1,\xi_1;\tau_2,\xi_2;\ldots;\tau_n,\xi_n)$ *of a reward process, for some fixed* n *, and define an "image" process by taking the variables in the reverse sequence* $(\tau_n,\xi_n;\tau_{n-1},\xi_{n-1};\ldots;\tau_1,\xi_1)$. *Any event defined by realizations of the original process has the same probability as the corresponding event defined by the image process.*

The term *reflection principle* arises from the fact that the image process is the one seen by someone who "stands" on the point (ρ_n,ς_n) of Figure 1, and reverses the orientation of both axes. (The rewards also appear as if earned at the beginning of the interval, but this is unimportant.)

Figure 3 shows an important application of the reflection principle to prove:

$$\begin{aligned}(13)\qquad & P\{\rho_n > \rho_{n-1}\ ,\ \rho_n > \rho_{n-2},\ \ldots,\ \rho_n > \rho_1\ ;\ 0 < \rho_n \le y\ ;\ \varsigma_n \le t\} = \\ & = P\{\rho_1 > 0\ ,\ \rho_2 > 0,\ \ldots,\ \rho_{n-1} > 0\ ;\ 0 < \rho_n \le y\ ;\ \varsigma_n \le t\}\end{aligned}\qquad \begin{pmatrix} n = 1,2,\ldots \\ 0 < y < \infty \\ 0 < t < \infty \end{pmatrix}$$

The left hand side of (13) is the probability that n is the index of *some* ladder point whose epoch is less than or equal to t , and whose height is less than or equal to y . This is the three-dimensional analogue of the renewal function, M , (mean counting function) of classical renewal theory, so we set

$$(14)\qquad M^{+}(n,t;y) = \text{LHS of (13)}$$

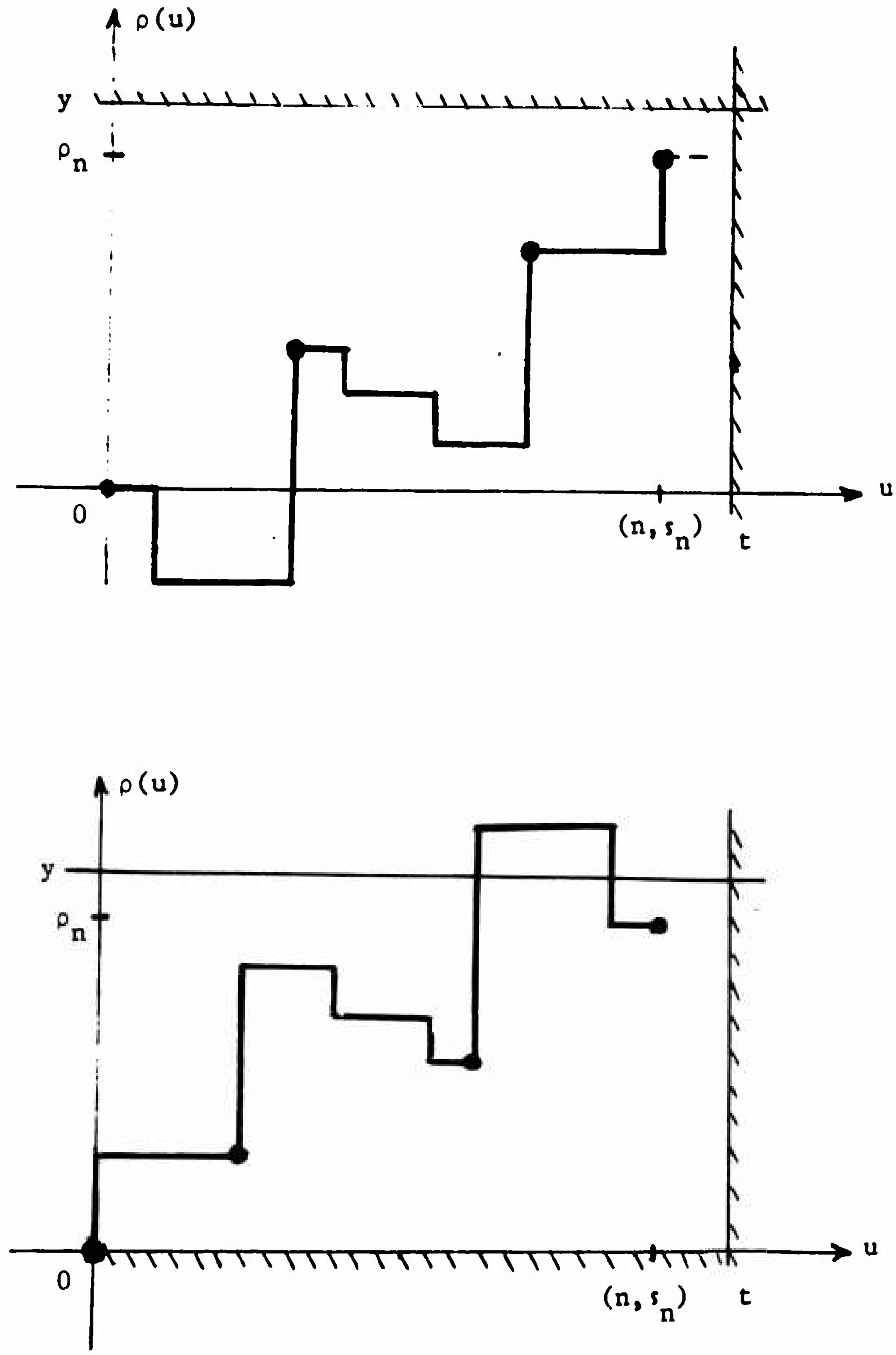

FIGURE 3. CORRESPONDING REALIZATIONS OF ORIGINAL AND IMAGE REWARD PROCESSES, SHOWING ORIGINAL LADDER POINTS.

Define three-fold transforms:

$$m^{+}(z,s;p) = \sum_{n=1}^{\infty} z^{n} \int_{0}^{\infty} e^{-st} \int_{0^{+}}^{\infty} e^{-py}\, d^{2}_{t,y} M^{+}(n,t;y)$$

(15) $$f^{+}(z,s;p) = \sum_{n=1}^{\infty} z^{n} \int_{0}^{\infty} e^{-st} \int_{0^{+}}^{\infty} e^{-py}\, d^{2}_{t,y} F^{+}(n,t;y) = E\{z^{\nu} e^{-s\phi} e^{-p\lambda}\} \quad ;$$

then from (10), and the fact that n must be the first *or* the second *or* the .... ladder point:

(16) $$m^{+} = f^{+} + (f^{+})^{2} + (f^{+})^{3} + \ldots = \frac{f^{+}}{1-f^{+}} \quad ,$$

in the appropriate region of the complex space (z,s,p) , analogous to the usual one-dimensional result in renewal theory.

The right hand side of (13), on the other hand, is a measure for the n^{th} epoch (not necessarily a ladder epoch), with $\mathfrak{s}_n \leq t$ and $\rho_n \leq y$, *for a process which has yet to reach its first weak descending ladder point* (excluding the origin) -- i.e., *has yet to go nonpositive* (excluding $[0,\mathfrak{s}_1])$. Thus:

(17)
$$P\{\nu^{\ominus}=1;\phi^{\ominus}\leq t\} = Q(t;0^{+}\}$$
$$P\{\nu^{\ominus}=n+1;\phi^{\ominus}\leq t\} = \int_{u=0}^{t}\int_{x=0^{+}}^{\infty} Q(t-u;x)\, d^{2}_{u,x} M^{+}(n,u;x) \qquad (n = 2,3, \ldots)$$

Similar duality relationships exist between F^{+} and $M^{\ominus}$, etc.

4. DECOMPOSITION OF DUAL PROCESSES

With this dual interpretation of M^{+} , its complete form, and that of the dual first-passage (weak descending ladder) distribution $F^{\ominus}$, can be expressed in

recursion relations in terms of the joint distribution Q. From the definitions:

$$(18)\qquad M^{+}(1,t;y) = Q(t;y) - Q(t;0)$$

$$\begin{pmatrix} 0 \le t < \infty \\ 0 < y < \infty \\ n = 1,2,3, \ldots \end{pmatrix}$$

$$(19)\qquad M^{+}(n+1,t;y) = \int_{u=o}^{t}\int_{x=o^{+}}^{\infty} [Q(t-u;y-x) - Q(t-u;-x)]\,d^{2}_{u,x}M^{+}(n,u;x)$$

which can be built up recursively. For the nonpositive values of y, we have simply

$$(20)\qquad F^{\ominus}(1,t;y) = Q(t;y)$$

$$\begin{pmatrix} 0 \le t < \infty \\ -\infty < y \le 0 \\ n = 1,2,3, \ldots \end{pmatrix}$$

$$(21)\qquad F^{\ominus}(n+1,t;y) = \int_{u=o}^{t}\int_{x=o^{+}}^{\infty} Q(t-u;y-x)\; d^{2}_{u,x}M^{+}(n,u;x)$$

Because of the nonoverlapping ranges of M^{+} and $F^{\ominus}$, except for the continuation

$$(22)\qquad F^{\ominus}(n,t;y) = F^{\ominus}(n,t;0)\ ,\ (y>0)$$

there is no ambiguity if (18) - (21) are written together as:

$$(23)\qquad M^{+}(1,t;y) + F^{\ominus}(1,t;y) = Q(t;y)$$

$$\begin{pmatrix} 0 \le t < \infty \\ -\infty < y < +\infty \\ n = 1,2, \ldots \end{pmatrix}$$

$$(24)\qquad M^{+}(n+1,t;y) + F^{\ominus}(n+1,t;y) = \int_{u=o}^{t}\int_{x=o^{+}}^{\infty} Q(t-u;y-x)\; d^{2}_{u,x}M^{+}(n,u;x)$$

Setting $t=\infty$ gives the usual representation of an ordinary fluctuation process [8].

Taking the transform of (23) - (24), using definitions similar to (15) and

$$(25)\qquad q(s;p) = \int_{0}^{\infty} e^{-st}\int_{-\infty}^{\infty} e^{-py}\, d^{2}_{t,y}Q(t;y) = E\{e^{-s\tau-p\xi}\}\ ,$$

we obtain the neater result:

$$m^{+}(z,s;p) + f^{\ominus}(z,s;p) = zq(s;p)[1+m^{+}(z,s;p)] \tag{26}$$

or

$$[1+m^{+}(z,s;p)]\,[1-zq(s;p)] = 1-f^{\ominus}(z,s;p)$$

from which the special cases can be easily deduced.

By setting $z = [1/q(s;p)]$ in (26) or (28) (which operation can be justified, see [8]), we obtain a generalization of *Wald's Identity*:

$$E\left\{\left(\frac{1}{q(s;p)}\right)^{\nu} e^{-s\phi}\, e^{-p\lambda}\right\} = 1 \tag{27}$$

which clearly holds for *all* types of ladder points (11).

5. WIENER-HOPF FACTORIZATION

If (16) is substituted in (26) we obtain the striking result

$$[1-f^{+}(z,s;p)]\,[1-f^{\ominus}(z,s;p)] = 1-zq(s;p) \quad , \tag{28}$$

that is,

$$[1-zE\{e^{-s\tau-p\xi}\}] = \left[1-E\left\{z^{\nu^{+}} e^{-s\phi^{+}-p\lambda^{+}}\right\}\right]\left[1-E\left\{z^{\nu^{\ominus}} e^{-s\phi^{\ominus}-p\lambda^{+}}\right\}\right] .$$

This decomposition of the honest distribution Q into two distributions, F^{+} and $F^{\ominus}$, one of which may be defective, is a *Wiener-Hopf factorization*; in many simple cases of interest, such as rational transforms, the factorization can be performed explicitly, using knowledge of the regions of analyticity in the complex p-plane which follow from the different domains of definition of F^{+} and $F^{\ominus}$. (See [8], [15], and [16] for examples of factorization.)

The asymmetry in (28) can be removed by defining a *distribution of first return to the origin* as the limit of $y \uparrow 0^{-}$ of $F^{\ominus}(n,t;y)$, viz:

$$F^{\odot}(n,t) = P\{\nu_1^{\odot}=n, \phi_1^{\odot}\le t\} = P\{\rho_1>0,\rho_2>0,\ \ldots,\ \rho_{n-1}>0;\rho_n=0,\varsigma_n\le t\} = \tag{29}$$

$$= P\{\rho_1<0,\rho_2<0,\ \ldots,\ \rho_{n-1}<0;\rho_n=0,\varsigma_n\le t\} \quad .$$

Taking transforms and using the obvious:

$$(1+m^{\ominus}) = (1+m^{\odot})\ (1+m^{-}) \quad , \tag{30}$$

the symmetric form of (28) is obtained

$$[1-f^{+}]\ [1-f^{-}]\ [1-f^{\odot}] = 1 - zq \quad ; \tag{31}$$

this distinction vanishes when $Q(y)$ is continuous.

6. THE CYCLIC PERMUTATION PRINCIPLE: THE EXPLICIT FACTORIZATION IDENTITY

The second basic combinatorial result is based upon the n possible cyclic permutations of the reward structure.

Define the event:

$$\mathcal{E}_r(n,t;y) = \{n \text{ is the } r^{th} \text{ ladder index; } \varsigma_n\le t;\rho_n\le y\} \quad . \quad \begin{pmatrix} n = 1,2,\ \ldots \\ t \ge 0 \\ y > 0 \end{pmatrix} \tag{32}$$

Then:

Cyclic Permutation Principle

$$\frac{1}{n}\,P\{\varsigma_n \le t\ ;\ 0 < \rho_n \le y\} = \sum_{r=1}^{\infty} \frac{1}{r}\,P\{\mathcal{E}_r(n,t;y)\} \tag{33}$$

Proof:

The proof of (33) follows exactly a proof of Feller for the case $t=\infty$, $y=\infty$ ([8], p. 395). Q.E.D.

The first probability in (33) is governed by the n-fold convolution of Q , and the probability on the RHS is the r-fold convolution of F^{+} . Hence, taking transforms over the appropriate range:

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \int_0^{\infty} e^{-st} \int_{0^+}^{\infty} e^{-py}\, d^2_{t,y} Q^{n*}(t;y) = \sum_{r=1}^{\infty} \frac{1}{r}\, [f^+(z,s;p)]^r =$$

$$= -\ln[1-f^+(z,s;p)] \quad ,$$

we obtain an *explicit factorization* of (28):

$$(34) \qquad f^+(z,s;p) = 1-e^{-\sum_{n=1}^{\infty} \frac{z^n}{n} \int_0^{\infty} e^{-st} \int_{0^+}^{\infty} e^{-py}\, d^2_{t,y} Q^{n*}(t;y)}$$

Various factorizations of this type were first obtained by Andersen, Pollaczek, and Spitzer, so it is perhaps appropriate to refer to (34) as an *Andersen-Pollaczek-Spitzer Factorization Identity*. (See discussion in [21] and [6].)

By analogy with previous results, (34) becomes an omnibus formula for *all* first-passage distributions according to the following scheme:

$$(35) \qquad \begin{aligned} f^+ &\longleftrightarrow \int_{0^+}^{\infty} \\ f^{\oplus} &\longleftrightarrow \int_{0^-}^{\infty} \\ f^{\odot} &\longleftrightarrow \int_{0^-}^{0^+} \\ f^{\ominus} &\longleftrightarrow \int_{-\infty}^{0^+} \\ f^- &\longleftrightarrow \int_{-\infty}^{0^-} \end{aligned} \quad ,$$

showing the appropriate range of y-integration.

In fact,

$$[1-f^{+}]\,[1-f^{\ominus}] = e^{-\sum \frac{z^n}{n} \int_0^\infty e^{-st}\left[\int_{0^+}^{\infty} + \int_{-\infty}^{0^+}\right] e^{-py}\, d^2_{y,t} Q^{n*}(t,y)} =$$

$$= e^{-\sum \frac{z^n}{n} [q(s,p)]^n} = 1-zq(s,p)$$

which provides independent verification of (28). Or, conversely, taking logarithms of (26), expanding $\ell n[1-zq(s,p)]$ in powers of z, and factoring in the complex p-plane will provide independent verification of the identity (34).

It should be noted that integration with respect to t does not require factorization in the above sense.

7. CLASSIFICATION OF BEHAVIOR

Because the reward process defined on the indices of the epochs is just the classic random walk in $\mathcal{R}^1$, the basic classification scheme for these fluctuation processes still must hold, with some slight modification in the case where the interval between epochs has infinite mean. We summarize these known results below, for completeness.

Assume for the moment that $F^{\ominus}$ is an honest distribution; from (34), or (20) - (21):

(36) $$E\{\nu^{\ominus}\} = \frac{1}{1-F^{+}_{\infty}} \quad ,$$

(37) $$E\{\phi^{\ominus}\} = \frac{E\{\tau\}}{1-F^{+}_{\infty}} \quad ,$$

and

(38) $$E\{\lambda^{\ominus}\} = \frac{E\{\xi\}}{1-F^{+}_{\infty}} \quad ,$$

with

$$F_\infty^+ = \sum_{j=1}^{\infty} F^+(j,\infty;\infty) \leq 1 \quad .$$

Similar relationships hold between any dual pairs of ladder processes. Note that (38) is meaningless unless $E\{\xi\} \leq 0$, and that all means except possibly (38) are infinite if F^+ is honest.

On the other hand, from (34) - (35)

$$F_\infty^+ = 1-e^{-S^+} \tag{39}$$

with

$$S^+ = \sum_{n=1}^{\infty} \frac{1}{n} [1-Q^{n*}(\infty;0)] = \sum_{n=1}^{\infty} \frac{1}{n} P\{\rho_n>0\} \quad , \tag{40}$$

and similarly for $F_\infty^\ominus$, which implies

$$S^\ominus = \sum_{n=1}^{\infty} \frac{1}{n} P\{\rho_n \leq 0\} \tag{41}$$

must diverge if $F^\ominus$ is honest.

Note that mean ladder epochs and heights may be infinite, in the honest cases, if $E\{\tau\}$ or $E\{\xi\}$ is infinite.

Additionally, if *both* ladder distributions are honest, it follows that $E\{\xi\} = 0$ (and vice versa), and then *all* ladder epochs and indices are infinite. For the mean ladder heights, however, differentiating (28) gives the well-known result:

$$E\{\lambda^+\} \cdot E\{-\lambda^\ominus\} = +\tfrac{1}{2}V\{\xi\} \quad , \tag{42}$$

so that the mean ladder heights are both finite, if the variance of ξ is finite.

In fact, using the Central Limit Theorem and a Tauberian theorem on (34), one can show the following explicit result, due to Spitzer (see [8]):

$$(43) \qquad E\{\lambda^+\} = +\sqrt{\frac{V\{\xi\}}{2}}\, e^{-C^+} \; ; \; E\{\lambda^-\} = -\sqrt{\frac{V\{\xi\}}{2}}\, e^{-C^{\ominus}} \; ,$$

where C^+ is the conditionally convergent series

$$(44) \qquad C^+ = \sum_{n=1}^{\infty} \frac{1}{n}\,[P\{\rho_n > 0\} - \tfrac{1}{2}] = -C^{\ominus} = -\sum_{n=1}^{\infty} \frac{1}{n}\,[P\{\rho_n \leq 0\} - \tfrac{1}{2}]$$

Summarizing these results for the dual pair $\{F^+, F^{\ominus}\}$, we have the classification scheme shown in Figure 4.

8. FURTHER IDENTITIES

If a ladder distribution is honest, and the required moments are finite, then continued manipulation of (26) will produce the following "Waldian" identities for the moments of the record values:

$$(45) \qquad E\{\nu\} = 1 + E\{\text{total number of ladder points in the dual ladder process}\} = e^{S^{\text{dual}}}$$

$$(46) \qquad E\{\phi\} = E\{\tau\} \cdot E\{\nu\}$$

$$(47) \qquad E\{\lambda\} = E\{\xi\} \cdot E\{\nu\}$$

$$(48) \qquad V\{\phi\} = 2E\{\tau\} \cdot E\{\nu\lambda\} + E\{\nu\} \cdot V\{\tau\} - [E(\tau)]^2 E\{\nu^2\}$$

$$(49) \qquad V\{\lambda\} = 2E\{\xi\} \cdot E\{\nu\phi\} + E\{\nu\} \cdot V\{\xi\} - [E(\xi)]^2 E\{\nu^2\}$$

$$(50) \qquad \begin{aligned} E\{\phi\lambda\} &= E\{\xi\} \cdot E\{\nu\lambda\} + E\{\tau\} \cdot E\{\nu\phi\} + E\{\nu\} \cdot E\{\xi\tau\} - \\ &\quad E\{\nu\} \cdot E\{\tau\} \cdot E\{\xi\} - E\{\tau\} \cdot E\{\xi\} \cdot E\{\nu^2\} \end{aligned} \; ,$$

	REWARD PROCESS	STRICT ASCENDING LADDER PROCESS					WEAK DESCENDING LADDER PROCESS				
TYPE OF PROCESS	$E\{\xi\}$	F_∞^+	S^+	$E\{\nu^+\}$	$E\{\phi^+\}$	$E\{\lambda^+\}$	$F_\infty^\ominus$	$S^\ominus$	$E\{\nu^\ominus\}$	$E\{\phi^\ominus\}$	$E\{\lambda^\ominus\}$
DRIFT TO $+\infty$	>0	1	∞	$e^{S^\ominus}$ $(<\infty)$	$E\{\tau\}e^{S^\ominus}$ $(\le\infty)$	$E\{\xi\}e^{S^\ominus}$ $(\le\infty)$	$1 - e^{-S^\ominus}$ (<1)	$<\infty$	—	—	—
DRIFT TO $-\infty$	<0	$1 - e^{-S^+}$ (<1)	$<\infty$	—	—	—	1	∞	e^{S^+} $(<\infty)$	$E\{\tau\}e^{S^+}$ $(\le\infty)$	$E\{\xi\}e^{S^+}$ $(\ge-\infty)$
OSCILLATING	$=0$	1	∞	∞	∞	$\sqrt{\frac{V\{\xi\}}{2}}\, e^{C^\ominus}$ $(\le\infty)$	1	∞	∞	∞	$-\sqrt{\frac{V\{\xi\}}{2}}\, e^{C^+}$ $(\ge-\infty)$

$$S^+ = \sum \frac{1}{n} P\{\rho_n > 0\} \qquad S^\ominus = \sum \frac{1}{n} P\{\rho_n \le 0\} \qquad S^+ + S^\ominus = \infty$$

$$C^+ = \sum \frac{1}{n} \left[P\{\rho_n > 0\} - \frac{1}{2}\right] \qquad C^\ominus = \sum \frac{1}{n} \left[P\{\rho_n \le 0\} - \frac{1}{2}\right] \qquad C^+ + C^\ominus = 0$$

FIGURE 4. CLASSIFICATION OF FLUCTUATION PROCESSES, AND RESULTING BEHAVIOR OF DUAL LADDER PROCESSES.

etc. (46) and (47) may be contrasted with the expressions for the means of the ordinary values (see, for example, [11])

(51) $$E\{s_n\} = E\{\tau\} \cdot n \quad ,$$

and

(52) $$E\{\rho_n\} = E\{\xi\} \cdot n$$

or

(53) $$\lim_{t\to\infty} \frac{E\{\rho(t)\}}{t} = \frac{E\{\xi\}}{E\{\tau\}} \quad .$$

9. EXTREME VALUES

As an example of other random variables in a fluctuation process which are of interest, consider the *extreme* values of reward attained during some fixed horizon. For instance, the *maximum over the mixed horizon* $(N;T)$, $\varepsilon^{+}_{N;T}$, is defined as:

(54) $$\varepsilon^{+}_{N;T} = \max_{0\leq k\leq N} \{\rho_k \mid s_k \leq T\} \quad . \qquad \begin{pmatrix} N = 0,1,2, \ldots \\ 0 \leq T \leq \infty \end{pmatrix}$$

From Figure 2, we see that this extremum process is just the upper envelope created by the strict ascending ladder points, including the origin, which occur in $[0,N]$ *and* $[0,T]$.

Defining

(55) κ = the first index at which the maximum in (54) is attained

we have

(56) s_κ = the first epoch at which the maximum in (54) is attained

and the identity $\varepsilon^{+}_{N;T} \equiv \rho_\kappa$.

From first principles:

(57)
$$E^{+}(k,t;y|N;T) = P\{\kappa=k, \mathfrak{s}_{\kappa}\le t;\rho_{\kappa}\le y|N;T\}$$
$$= [1-F^{+}(N-k;T-t;\infty)][\delta_{ko}1(t)1(y)+M^{+}(k,t;y)] \quad , \quad \begin{pmatrix} 0 \le k \le N \\ 0 \le t \le T \\ 0 \le y \le \infty \\ N = 0,1,2, \ldots \\ 0 \le T \le \infty \end{pmatrix}$$

with $\delta_{oo} = 1$, $\delta_{ko} = 0$ $(k \neq o)$, and $1(t)$ as the unit step.

Using (10) and (34), the five-fold transform of (57) is:

(58)
$$e^{+}(z,s;p|w;q) = \sum_{N=o}^{\infty} w^{N} \int_{o}^{\infty} e^{-qT} E\left\{z^{\kappa} e^{-s\mathfrak{s}_{\kappa}} e^{-p\rho_{\kappa}}\right\} dT$$
$$= \frac{e^{-\sum_{n=1}^{\infty} \frac{w^{n}}{n} \int_{t=o}^{\infty} e^{-qt} \int_{y=o^{+}}^{\infty} [1-z^{n}e^{-st}e^{-py}] d^{2}_{t,y} Q^{n*}(t;y)}}{q[1-w]} \quad ,$$

from which many previously known special cases can be derived. (See, for example, Chapter 6, [16] or Chapter 18, [8].)

In particular, we mention the generalization of well-known formulas for the mean index, epoch, and value of the maximum attained in $(N;T)$:

(59)
$$E\{\kappa \mid N;T\} = \sum_{n=1}^{N} \int_{t=o}^{T} \int_{y=o^{+}}^{\infty} d^{2}_{t,y} Q^{n*}(t;y)$$

(60)
$$E\{\mathfrak{s}_{\kappa} \mid N;T\} = \sum_{n=1}^{N} \frac{1}{n} \int_{t=o}^{T} t \int_{y=o^{+}}^{\infty} d^{2}_{t,y} Q^{n*}(t;y)$$

(61)
$$E\{\rho_{\kappa} \mid N;T\} = \sum_{n=1}^{N} \frac{1}{n} \int_{t=o}^{T} \int_{y=o^{+}}^{\infty} y \, d^{2}_{t,y} Q^{n*}(t;y) \quad .$$

Because of its intimate relationship with the strict ascending ladder process, the maximum process also drifts off to $+\infty$, as N and T increase without limit, *unless* $E\{\xi\} < 0$. In this case κ is finite, with probability one, and we have simply:

$$(62)\qquad E\left\{z^{\kappa} e^{-s\,\mathfrak{s}_{\kappa}} e^{-p\rho_{\kappa}}\right\} = e^{-\sum_{n=1}^{\infty} \frac{1}{n} \int_{t=o}^{\infty} \int_{y=o^{+}}^{\infty} [1-z^{n}e^{-st}e^{-py}]d^{2}_{t,y}Q^{n*}(t;y)}$$

$(N=\infty, T=\infty; E\{\xi\}<0)$. $E\{\kappa|\infty,\infty\}$ is finite, but the magnitudes of $E\{\mathfrak{s}_{\kappa}|\infty,\infty\}$ and $E\{\rho_{\kappa}|\infty,\infty\}$ depend upon further assumption about Q. By comparison with (34) and (39), we also see that in the infinite-horizon case:

$$(63)\qquad E^{+}(k,t;y|\infty;\infty) = [1-F^{+}_{\infty}][\delta_{ko}1(t)1(y)+M^{+}(k,t;y)] \quad ,$$

which could have been predicted from first principles.

Another interesting case occurs if $Q(t,y)$ *is symmetric about* $y=o$ *for all values of* t. Then, by direct summation of (62), the index and epoch of the maximum have the transform:

$$(64)\qquad e^{+}(z,s;o|w;q) = \frac{\sqrt{1-wa(q)}}{q(1-w)\sqrt{1-wza(s+q)}}$$

where $a(s) = E\{e^{-s\tau}\}$. No finite maximum exists for this oscillating case as N and $T \to \infty$. However, the *normalized* variables

$$(65)\qquad \alpha = \frac{\kappa}{N} \quad ; \quad \beta = \frac{\mathfrak{s}_{\kappa}}{T}$$

have a joint density whose mass lies entirely along the line $\alpha = \beta$ $(0\leq\alpha\leq1)$. The marginal distribution of *either* α or β is:

$$(66)\qquad P\{\alpha \leq x\} = \frac{2}{\pi}\sin^{-1}\sqrt{x} \qquad 0 \leq x \leq 1 \quad ,$$

the *arc-sine law* so well-beloved of Feller [8]. Similar results hold for the joint distribution of the maximum and $\rho_{\min[N,n(T)]}$, or the joint distribution of the maximum and minimum in (N;T) (see [8] and [16] for corresponding results in ordinary fluctuation theory.)

10. GI/G/1

To illustrate an application of renewal-reward fluctuations, consider the GI/G/1 queue. It is a well-known result from queueing theory that, if customers numbered 0,1,2, ... arrive at epochs $\tau_o = 0, \tau_1, \tau_1+\tau_2, \ldots$ and have service times $\sigma_o, \sigma_1, \sigma_2, \ldots$, then the *delay in queue of the* n^{th} *customer*, δ_n , is given by the recursion relationship:

$$\begin{aligned} &\delta_o = 0 \\ &\delta_n = [\delta_{n-1} + \sigma_{n-1} - \tau_n]^+ , \qquad n = 1,2, \ldots \end{aligned} \tag{67}$$

where $[x]^+ = \max(x,o)$, and it is assumed there were no customers present at the zeroth arrival.

If we identify the rewards of the fluctuation process as:

$$\begin{aligned} &\xi_o = 0 \\ &\xi_n = \sigma_{n-1} - \tau_n , \qquad n = 1,2, \ldots \end{aligned} \tag{68}$$

then we see that $\rho_n = \sum_{i=0}^{n} \xi_i$ is identical with δ_n *as long as* δ_n *stays positive*. If

$$\pi^{\ominus}(n) = \max_{j=0,1,\ldots,n} \left\{ \nu_o^{\ominus} + \nu_1^{\ominus} + \ldots \nu_j^{\ominus} \leq n \right\} \tag{69}$$

is the last weak descending ladder index (including zero), then, in general,

$$\delta_n = \rho_n - \rho_{\pi^{\ominus}(n)} . \tag{70}$$

(See Chapter 6, [8], and Chapter 4, [15].)

For a GI/G/1 queueing process which begins with no customers present at the zeroth arrival, define:

(71)
$$\begin{aligned} \nu &= \text{the number of customers served in a busy period} \\ \beta &= \text{the duration of a busy period} \\ \gamma &= \text{the duration of a busy cycle} \end{aligned}$$

and

$$\iota = \gamma - \beta = \text{the duration of an idle period}$$

Then it follows directly from the above observations that:

(72)
$$\begin{aligned} \nu &\sim \nu^{\ominus} \\ \iota &\sim -\lambda^{\ominus} \end{aligned}$$

if the reward structure is chosen from (68) ($\sim$ means "equal in distribution"). If the underlying renewal process (2) is then chosen to be identical with the queueing arrival process, then:

(73)
$$\begin{aligned} \gamma &\sim \phi^{\ominus} \\ \beta &\sim \phi^{\ominus} + \lambda^{\ominus} \end{aligned} ,$$

and we can use previous results, especially the appropriate form of (34) - (35).

If

(74)
$$\begin{aligned} P\{\sigma_i \le t\} &= S(t) \quad ; \\ P\{\tau_i \le t\} &= A(t) \quad ; \end{aligned} \quad \begin{pmatrix} i = 0,1,2, \ldots & \\ & t \ge 0 \\ i = 1,2, \ldots & \end{pmatrix}$$

then the appropriate form of Q is given by:

(75)
$$Q(y,t) = \int_0^t S(y+u)\, dA(u) \quad , \quad \begin{pmatrix} -t < y < \infty \\ t \ge 0 \end{pmatrix}$$

or, using lower-case letters for densities:

$$d^2_{y,t}Q(y,t) = s(y+t)\ a(t)\ dydt$$

and

$$d^2_{y,t}Q^{n*}(y,t) = s^{n*}(y+t)\ a^{n*}(t)\ dydt \qquad (n = 1,2,\ \ldots) \tag{76}$$

over the appropriate range.

Substituting in (34) - (35):

$$E\{z^{\nu}e^{-s\gamma}e^{+p\iota}\} = 1-e^{-\sum_{n=1}^{\infty}\frac{z^n}{n}\int_0^{\infty}e^{-(s-p)t}\,dA^{n*}(t)\int_0^t e^{-py}\,dS^{n*}(y)}, \tag{77}$$

which is implicit in a slightly more general formula due to Kingman ([12], p.351).

From the definition $\beta = \gamma - \iota$, an alternate form of (77) is obtained:

$$E\{z^{\nu}e^{-s\beta}e^{+p\iota}\} = 1-e^{-\sum\frac{z^n}{n}\int_0^{\infty}e^{pt}\,dA^{n*}(t)\int_0^t e^{-(s+p)y}\,dS^{n*}(y)}. \tag{78}$$

Let

$$p_n = P\{\rho_n > 0\} = \int_0^{\infty}[1-S^{n*}(t)]dA^{n*}(t)\quad . \tag{79}$$

Then for the busy cycle to be a recurrent event, i.e., for the busy period to terminate with probability one, $F^{\ominus}$ must be an honest distribution.

$$E\{\xi\} = E\{\sigma\} - E\{\tau\} \leq 0 \quad \text{or} \quad \frac{E\{\sigma\}}{E\{\tau\}} \leq 1 \tag{80}$$

is the usual queueing criterion; from (41), we see that an alternate criterion is:

$$\sum_{n=1}^{\infty} \frac{1}{n} [1-p_n] = \infty \quad . \tag{81}$$

From (78) and (79) follow the well-known results:

$$E\{\nu\} = e^{\sum_{n=1}^{\infty} \frac{1}{n} p_n} \quad ; \quad E\{\beta\} = E\{\nu\} \cdot E\{\sigma\} \quad ; \quad E\{\gamma\} = E\{\nu\} \cdot E\{\tau\} \quad ;$$

(82)

$$E\{\nu^2\} = E\{\nu\}\left[2 \sum_{n=1}^{\infty} p_n + 1\right] \quad ; \text{ etc.}$$

Other results can be expressed in terms of the p_n, or incomplete integral forms of (79), [10].

To obtain information about customer delay, let:

(83) $\delta(N;T) = \delta_{\min\,[N,\mathfrak{s}(T)]}$ = delay in queue of the last customer whose index is less than or equal to N, and whose arrival epoch is less than or equal to T.

Then, by a direct application of the reflection principle it follows from (70) that

$$\delta(N;T) \sim \varepsilon^{+}_{N;T} \tag{84}$$

if the correspondence (75) is followed. Direct results then can be obtained from (58) - (63). κ and $\mathfrak{s}_\kappa$ in (55) and (56) are then the index and arrival epoch of of the last customer in (83), *measured from the beginning of the busy period in progress.*

If conditions (80) are satisfied with a strict inequality, then a stationary delay distribution exists as $N \to \infty$ and $T \to \infty$, given by the appropriate form of (62):

(85) $$E\left\{z^{\kappa} e^{-s\varsigma_{\kappa}} e^{-p\delta}\right\} = \frac{1}{E\{\nu\}} e^{\sum_{n=1}^{\infty} \frac{z^n}{n} \int_0^{\infty} e^{-(s-p)t} dA^{n*}(t) \int_t^{\infty} e^{-py} dS^{n*}(y)} \quad .$$

Or, substitution of (63) into (18) - (19) gives a Wiener-Hopf integral equation which is a generalization of a famous result due to Lindley [13].

11. AN EXTENSION

It should be clear from the development that the addition of the renewal "random clock" created very little additional labor, since keeping track of the $\{\tau_i\}$ was unrelated to the decomposition which was made on the $\{\xi_i\}$.

Similarly, one may generalize these results to include an r-dimensional *vector* with components:

(86) $$\tau_i = \left\{\tau_i^1, \tau_i^2, \ldots, \tau_i^r\right\} \qquad (i = 1,2, \ldots)$$

with distribution

(87) $$Q(\mathbf{t};y) = P\{\boldsymbol{\tau}_i \leq \mathbf{t}; \xi_i \leq y\} \qquad \left(\begin{array}{c} i = 1,2, \ldots \\ \mathbf{t} \geq 0 \\ -\infty < y < +\infty \end{array}\right)$$

In fact, the components of the $\{\boldsymbol{\tau}_i\}$ do not have to be nonnegative if the fluctuations of the $\{\varsigma_i\}$ are not under investigation, i.e., if only $\{\rho_i\}$ is decomposed into dual ladder processes.

Almost all of the previous development carries through with obvious modification; in particular, if r-dimensional vectors of "epochs," $\boldsymbol{\phi}$, and transform variables, $\boldsymbol{s}$, are defined in the obvious way, then (34) becomes:

(88) $$E\{z^{\nu} e^{-\boldsymbol{s}\cdot\boldsymbol{\phi}} e^{-p\lambda}\} = 1-e^{-\sum \frac{z^n}{n} \iint_{T}\cdots\int e^{-\boldsymbol{s}\cdot\mathbf{t}} \int_{Y} e^{-py} d_y\left[d_{\mathbf{t}}^r Q^{n*}(\mathbf{t};y)\right]} \quad ,$$

with the range of integration Y being chosen for the appropriate ladder process, and T being chosen from the appropriate domain of the $\{\tau_i\}$.

Exploitation of (88) would appear to hold much promise for further results in queueing theory. Formal justification of the manipulations necessary could be made by appealing to powerful operator identities in Banach Algebra developed by Wendel [22,23] and Baxter [4,5,6].

REFERENCES

[1] Andersen, E. Sparre, "On Sums of Symmetrically Dependent Random Variables," Skand.Aktuar., Vol. 36, pp.123-138, (1953a).

[2] Andersen, E. Sparre, "On the Fluctuations of Sums of Random Variables I," Math.Scand., Vol. 1, pp.263-285, (1953b).

[3] Andersen, E. Sparre, "On the Fluctuations of Sums of Random Variables II," Math.Scand., Vol. 2, pp.195-223, (1954).

[4] Baxter, G., "An Operator Identity," Pac.Jour.Math., Vol. 8, No. 4, pp.649-663, (Winter 1958).

[5] Baxter, G., "A Two-Dimensional Operator Identity with Application to the Change of Sign in Sums of Random Variables," Trans.Amer.Math.Soc., Vol. 96, pp.210-221, (1960).

[6] Baxter, G., "An Analytic Approach to Finite Fluctuation Problems in Probability," Jour. d'Analyse Math., Vol. 9, pp.31-69, (1961).

[7] Blackwell, D., "Extension of a Renewal Theorem," Pac.Jour.Math., Vol. 3, pp.315-320, (1953).

[8] Feller, W., AN INTRODUCTION TO PROBABILITY THEORY AND ITS APPLICATION, Vol. II, J. Wiley & Sons, New York, 626 pp., (1966).

[9] Feller, W., "On Combinatorial Methods in Fluctuation Theory," Probability & Statistics, (The Harald Cramer Volume), J. Wiley & Sons, New York, pp.75-91, (1959).

[10] Finch, P. D., "On the Busy Period in the Queueing System GI/G/1," Jour. Australian Math.Soc., Vol. 2, No. 2, pp.217-228, (October 1961).

[11] Jewell, W. S., "Markov-Renewal Programming I & II," Operations Research, Vol. 11, No. 6, pp.938-971, (November-December 1963).

[12] Kingman, J.F.C., "The Use of Spitzer's Identity in the Investigation of the Busy Period and Other Quantities in the Queue GI/G/1," Jour. Australian Math.Soc., Vol. 2, No. 3, pp.345-356, (February 1962).

[13] Lindley, D. V., "Theory of Queues with a Single Server," Proc.Cam.Phil.Soc., Vol. 48, pp.277-289, (1952).

[14] Port, S. C., "An Elementary Probability Approach to Fluctuation Theory," Jour. Math.Anal.Appl., Vol. 6, No. 1, pp.109-151, (February 1963).

[15] Prabhu, N. U., QUEUES AND INVENTORIES, J. Wiley & Sons, New York, 275 pp., (1965).

[16] Prabhu, N. U., STOCHASTIC PROCESSES, The MacMillan Company, New York, 233 pp., (1965).

[17] Spitzer, F., "A Combinatorial Lemma and Its Applications to Probability Theory," Trans.Amer.Math.Soc., Vol. 82, pp.323-339, (1956).

[18] Spitzer, F., "The Wiener-Hopf Equation Whose Kernel is a Probability Density," Duke Math.J., Vol. 24, pp.327-344, (1957).

[19] Spitzer, F., "The Wiener-Hopf Equation Whose Kernel is a Probability Density II," Duke Math.J., Vol. 27, pp.363-372, (1960a).

[20] Spitzer, F., "A Tauberian Theorem and Its Probability Interpretation," Trans. Amer.Math.Soc., Vol. 94, pp.150-160, (1960b).

[21] Syski, R., "Pollaczek Method in Queueing Theory," NATO Conference on Recent Developments of Queueing Theory and Its Applications, Lisbon, Portugal, (September 1965).

[22] Wendel, J. G., "Spitzer's Formula: A Short Proof," Proc.Amer.Math.Soc., Vol. 9, pp.905-908, (1958).

[23] Wendel, J. G., "Order Statistics of Partial Sums," Ann.Math.Stat., Vol. 31, No. 4, pp.1034-1044, (December 1960).

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13 ABSTRACT

Fluctuation theory is conerned with the study of extreme values of sums of independent, arbitrary-valued random variables. Simple but powerful combinatorial methods due chiefly to E. S. Andersen, F. Spitzer, and W. Feller have recently provided an easy method of attack on these problems. However, operations research models are concerned with fluctuations of various economic returns which are earned at random points in time, and whose increments are correlated with the interval since the last payoff. Our generalization considers the fluctuations of a cumulative reward process, defined on an underlying renewal process. Most of the classical results carry through, including Weiner-Hopf type factorization, an Andersen-Pollaczek-Spitzer type identity, and certain Waldian-Pollaczek results. As applications, we find the distribution of the maximum return over a mixed index-epoch horizon, and show how certain general results for the GI/G/1 queue follow directly from the various three-dimensional ladder distributions.

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